Δ Calculus · Notes — April 13, 2020

Some notes on **continuity**, inspired by Jun's notes of April 9, and the Fan Alchemist discussion that day. Specifically, what I start to explore here is the possibility of defining regions or intervals first (as primitive), and then singletons (points, numbers, etc.) as limits of regions as their extent diminishes towards zero.

At a minimum, this shows that I have far more questions than ideas. More seriously, I can't yet say whether I think this is going to work. Hmmm…

I • Project

A. Continuity is normally dealt with via zero-dimensional *points* (numbers, typically), plus limits expressed in terms of sets of those points (e.g., limit $x \to 0$ sin (x)).

B. Conceptually, it seems as if discussions of continuity involve *intervals* or *regions* (assuming that an interval is a one-dimensional region).

   1. The standard δ/ε framing considers a small interval around both x and f(x)—a region of size ε in the f(x) value, δ in the x value.
   2. Integrals are often over regions or intervals ($\int_{x/0\to1}$, e^x)
      a. Provided there are coordinates? Not sure…

C. What would it be for the fan calculus (ΔC) to invert the explanatory order?

   1. Start with **regions** (as fundamental, primitive, whatever)
   2. Define **intervals** as one-dimension regions (but what does 'one-dimensional' mean?)
   3. Define **points** (numbers) as *limits of intervals*, as their extent reduces to nothing.
      a. Maybe points can be defined as limits of regions, too, as their total extent diminishes to zero? I.e., think of a patch of ground, and shrinking it continuously until it becomes a point—it is not clear that doing that requires identifying anything one-dimensional.

D. Questions

   1. Can this be done? (Has it been done? Whom could we ask?)
   2. What would the primitive (fundamental, kernel) properties of intervals be?
   3. What would the ΔC look like, if based on this idea?

II • Can this be done?

A. Tricky

   1. If one imagines a region or interval—or (easier) a region or interval *of something* (of a function, of a string, of a sheet of rubber)—it is natural to imagine that that region or interval has a *value* or *magnitude* everywhere along it. And natural to think of that value or magnitude as a *number* (i.e., a singleton).
   2. It seems as if the project requires taking the value or magnitude *itself* to be an interval—not, at least in the first instance, to be a number or point. What would that mean?

B. Hope

   1. Maybe two things make that idea palatable:
      a. Epistemically: in the real world (i.e., outside "pure mathematics"), one never knows, exactly, what any given magnitude is. All one can know is that any actual magnitude is *within an interval* (as we say: "2.74 ± 0.4," etc.)
         i. Recursively, the end points of the region are also never themselves perfectly known. Nor are the extents to which they are not known perfectly known, either. I've

wondered about this in the past: what would it be to say that the value of q is 2.74 ± (0.4 ± (0.02 ± (…))), recursively, forever…

b. Ontologically: Framed traditionally, the idea that an interval or region τ has a value or magnitude everywhere within it may just be shorthand for thinking that τ has a value or magnitude at every *point* within it. But that uses the idea of a point. So maybe instead the intuition should be recast: that, for every sub-region σ of τ, there will be '*a* value or magnitude' of that sub-region, where the "value or magnitude" is *itself* an interval—i.e., (I presume) the range of values or magnitudes that are exemplified along or within σ.

   i. Question: if σ is a sub-region of τ, and its value/magnitude is region m_σ, should it be assumed that there is what I have called a "**point-to-point correspondence** (PPC)" between σ and m_σ? This is tricky. Point-to-point correspondence is crucial to me in other contexts (deixis, consciousness). The question is whether one can make sense of it without needing a notion of a singleton point. Maybe. Intuitively, one can lay one strip along another, setting up something close to PPC, without necessarily needing to think about individual points (except in regards to whether the two strips line up perfectly—though of course practically one can line them up only "to within an interval" of accuracy ;-)).

   ii. This is a serious question. I don't know what to think about it right now.

2. This suggests what may be the crucial insight. Get at it in stages:

   a. First pass (untenable): whenever we talk about any singleton item in the classical view—i.e., whenever we use a singular term ("the value," "the length of this piece of wood," "the square root of 16", etc.) what we are really doing is referring to an interval (a one-dimensional region, in each of these cases). So we should say that—change our language always to talk about intervals.

      This doesn't work:

      α. Problem 1: if *everything* is an interval, doesn't the idea of 'interval' becomes vacuous?

      β. Problem 2: there is no way this will ever ground out: if intervals have properties such as extents and "ends," then those will be intervals, etc., etc., ad infinitum.

   b. Second pass: Bite the bullet (and recognize that the first pass is reactionary).

      i. It is not that points, numbers, etc., are not singletons. They *are* singletons. We need to accept the traditional view in this sense: *talking about* (registering) *singletons is how we should keep speaking* (registering). What changes is *what we think singletons are*.

      ii. In particular, recognize (a good ΔC tenet) that:

         α. Singletons—points, numbers, magnitudes, lengths, etc.—*are* (or can be considered to be) *apices,*[1] which (like all Δs) can be *opened up*.

         β. I.e., *a region or interval* is an "opened up singleton" (point).

         — Rather, in terms of definitions, singletons are *closed regions*.

         — What has been opened up? The interval, of course. Is that fatuous?

         χ. Intervals will (in general? always?) have extents and boundaries—which, from the natural point of view of regarding the interval (i.e., regarding it *as an interval*), will be singletons. But, like everything, those extents and boundaries can be opened up

[1]'Apices' is recognized as a plural form of 'apex' (e.g., in Merriam Webster and other good dictionaries).

too. Yes, ad infinitum—but that is not a problem (it says here; cf. the 3Lisp reflective tower ;-)).

III • Terminology

A. Take Δ*x* to mean "an opening up of *x*"—i.e., a fan with x as apex.
  1. When any Δ is opened, there needs to be a criterion of differentiation. Call it ξ (need a better term): that which differentiates the elements on the edge of the fan.[2]
     a. So to consider 'help' as a *word-type* is to say that 'help' is a singleton from some points of view (the point of view of word-types!), but it can be opened up into different tokens, by a criterion ξ of *spatial occurrence*, or something like that. (Token-*uses*, as in Perry's deaf mute example, would be a further opening up—either directly from the word-type, or more indirectly by opening-up a token.)
  2. I will assume that opening up a real number will mean opening it up "along the real line" (not sure how that becomes automatic).
  3. Notation
     a. The bare term "*x*" will be taken to be a reference to *x as a singleton* (i.e., as a point)
        i. Is "x" a reference to x *as an apex*? Probably not, though we could decide one way or other, but 'apex' may make sense as a label only if x has been opened up.
     b. The term "Δ*x*" will be taken to be a reference to *x as an interval* (around x, presumably)
     c. In prior notes, I have used the following notations (are these too ambiguous?)
        i. Δx:ξ, to mean "x opened up according to differentiation criterion ξ".
        ii. x:ζ to mean "x as a ζ".
  4. I suppose these would nest. I.e., 'help':(Δword-type:spatial-instance) would refer to a token of the word 'help'?
  5. Intervals
     a. Not all fans open up into regions or intervals. The set of tokens of the word type 'help' isn't an *interval*, in particular. Similarly, sets can be opened up to their elements, presumably; but the opening-up of a set isn't a region or interval, unless certain additional constraints are met. So something additional is needed, to be an interval or region.
     b. Think about intervals, first (simpler)
     c. It seems that if we are going to take a real number (or a measurement expressed as a real number plus unit, say) to be openable up into an interval, then we assume (this is totally a first pass!):
        i. A <u>criterion of differentiation</u> ξ (what makes different items in the interval distinct)
        ii. <u>Endpoints</u> (each specified as a singleton?)
            α. Or an extent, which could default to ±(extent/2) around the distinguished singleton (below)—or be calculated form the endpoints?)
        iii. A <u>distinguished singleton</u> *on the edge*, which is the exact value of the apex, as it were (so if 2.73 is a measurement, and it opens up to an interval 2.73 ± 0.04, then 2.7300, as it were, is the *distinguished singleton*. (⇐ I'm not at all sure about this).
        iv. An <u>ordering relation</u>

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[2] Is 'edge' a good term for the side of the triangle opposite the apex? Would 'fringe' be better? Maybe not (it sounds marginalizing).

d. Is there an assumption of density (countable? non-countable? it seems that both should be allowed)? Or would those regions (intervals) be called “dense regions,” etc.?
e. Perhaps there could be a type of differentiation criterion (call it ‘interval’) that builds these requirements in—i.e., such that any apex that is opened up according to a differentiation criterion that is itself an instance of ‘interval’ will mean that the items on the edge form an interval or region, with the above properties?
f. … Hmmm; not sure what to think about this…

IV • The differential calculus

A. Can we understand the traditional (differential) calculus from this point of view?
B. Continuity
1. Intuitively (and traditionally), it seems that an interval Δp is continuous if every “point” q (i.e., everything taken as a singleton) “between” p and any other point r ∈ Δp is itself ∈ Δp, using ‘∈’ to mean “within the region”, and according to ordering relation “between”.
   a. I.e.: [∀q ∈ ?? [∀r ∈ Δp [[r ≤ q ≤ p] ⊃ q ∈ Δp]]]
   b. But there is a question of where the q’s are selected from? It can’t be from Δp, because then the criterion would be satisfied vacuously. Hmm…
2. A function *f* is continuous at b if ∀x ∈ Δb [*f*(x) ∈ Δ*f*(b)].
   a. Surely this should be written as *f*(Δb) ∈ Δ*f*(b)—i.e., the function applied to a region around b should be mapped to a region around the function applied to b—where
      i. Δb is constrained by δ, in the standard sense; and
      ii. Δf(b) is constrained by ε?
   b. This is phrased in terms of a singleton (b). Is that OK?
   c. If $\Delta_y b$ is understood to be the interval [b-y, b+y] (or {b-y, b+y}—i am not sure about open and closed intervals yet, so maybe I will use $<\sigma_1, \sigma_2>$ for intervals from σ_1 to σ_2, to duck the question), then maybe *f* being continuous would be written as
      i. $[\forall\varepsilon, \exists\delta \mid [f(\Delta_\delta b) - f(b)] < \varepsilon]$
   d. Is there any chance that this could be written:
      i. $[\forall\varepsilon, \exists\delta \mid [f(\Delta_\delta b) - \Delta_\varepsilon f(b)] \rightarrow 0$
      ii. I don’t even know what that means ;-). Does it mean: $\lim_{\delta \rightarrow 0} f(\Delta_\delta b) = f(b)$?
3. What about the notions Jun highlights: set-closedness, preservation of continuity, and ratio-definiteness? Ratio-definiteness is required for derivatives. But I don’t know what it is to divide intervals; see §V, below.

C. Lots to think about…

V • Miscellaneous issues

A. What do arithmetic operations (‘+’, ‘–’, etc.) mean over intervals?
1. Intuitively, one could say this: close the two intervals (i.e., fan up to their apex), and do the standard arithmetic operation on the apices.
2. But it is not clear what the apex is if the fan hasn’t been created as the fan-out of a singleton (i.e. if there isn’t a distinguished singleton). What singleton should be chosen?
3. Surely the answer should be something like this (assume σ and τ are intervals), and that σ_{min} and σ_{max} are the minimum and maximum in interval σ, and similarly for τ.
   a. Addition/subtraction
      i. Addition and subtraction are maybe straightforward:

α. Addition: $\sigma + \tau \equiv <(\sigma_{min} + \tau_{min}), (\sigma_{max} + \tau_{max})>$
β. Subtraction: $\sigma - \tau \equiv <(\sigma_{min} - \tau_{max}), (\sigma_{max} - \tau_{min}\}>$

ii. These could be characterized in traditional terms, assuming that σ and τ are sets:
α. Addition: $\sigma + \tau \equiv \forall x \in \sigma, \forall y \in \tau \{z \mid z = x + y\}$
β. Subtraction: $\sigma - \tau \equiv \forall x \in \sigma, \forall y \in \tau \{z \mid z = x - y\}$

b. Multiplication/division
i. Multiplication and division are harder
ii. Traditional characterizations don't seem so difficult:
α. Multiplication: $\forall x \in \sigma, \forall y \in \tau \{z \mid z = x * y\}$
β. Division: $\forall x \in \sigma, \forall y \in \tau \{z \mid z = x / y\}$
iii. But I don't think they can be defined in terms of the limits or endpoints of σ or τ. If Δτ contains 4, even if τ itself is not 4, and the division is, as it were, …/(4–τ′) for some τ′ ∈ Δτ, then the maximum could be arbitrarily large.
iv. Similarly, there is an issue of multiplication—if the min of σ and τ are both negative numbers, their product will be positive, and so could be a maximum.
v. But maybe the arithmetic of negative numbers is peculiar.

c. Do topologists have a way to define all of these things?